BERICHTE

Some Topics in the Theory and Applications of the Optical Model

M. Bertero and G. Passatore

Istituto di Scienze Fisiche dell'Università di Genova, Genova, Italy

To Professor Konrad Bleuler *on the occasion of his 60th birthday*

In some recent works a dispersion relation proposed by Feshbach for the generalized optical potential has been used in order to analyse the energy dependence of the strength of the empirical optical potential. In the first part of this report the mathematical problems arising in the derivation of the dispersion relation and the results obtained in this direction are discussed; in the second part the approximations introduced in the application of the dispersion relation are analysed and the results concerning the non-locality of the generalized optical potential are reviewed.

1. Introduction

In the past few years many general features of the optical model for the scattering of nucleons by nuclei have been analysed and clarified.

A basic point is the following: the empirical (local) optical potential introduced to describe the total and differential cross sections has to be considered as related to a generalized optical potential which, in principle, can be derived from the dynamical equations for a many-nucleon system.

This point of view leads to theoretical problems concerning the definition of the generalized optical potential and the analysis of its properties and of its connection with the empirical optical potential.

If the energy of the incident nucleon is non-relativistic, it is quite natural to assume that the nucleon-nucleus system is described by a Schrödinger equation containing an interaction between nucleons in terms of two-body local potentials. In such a case one has a well defined mathematical model and one can define a generalized optical potential by projecting the scattering wave-function on the elastic channel [1-4]. Then one finds that such a potential is nonlocal, energy dependent and complex (in the energy range where inelastic channels are open). Besides, it can be analytically continued in the complex energy plane and it satisfies a dispersion relation which connects its real and imaginary parts.

The previous mathematical scheme is inadequate for two reasons: the first one is that high-energy nucleon-nucleus scattering is not considered; the second one is that, if one analyses at low energies the optical potential by means of the dispersion relation, one needs a knowledge of the asymptotic behaviour of the imaginary part of the generalized optical potential for large (relativistic) values of the energy. However in the relativistic case the problem of the meaning and of the existence of a generalized potential has not yet been solved. One can assume that the generalized potential has a very general meaning as suggested by its derivation, in the non-relativistic case, through a projection of the scattering wave function on the elastic channel, whatever the nature of the inelastic channels; but the validity of this point of view is not proved. A derivation of the generalized optical potential in the case of simple field models may give some insight into the general features of the problem. At present the optical model is used in relativistic situations with modifications only in the kinematics [5].

A second basic problem, which concerns the applications of the optical model is the following: since the generalized optical potential is nonlocal whereas the empirical optical potential is local, the dispersion relation between the real and imaginary parts of the generalized potential does not imply a dispersion relation between the real and imaginary parts of the empirical potential. Therefore, as already suggested by Feshbach, deviations of the empirical potential from the dispersion relation would

indicate the extent of the nonlocal nature of the generalized potential.

Since all the representations of the generalized potential are very involved, the previous program can be developed in a clear way only in the limit case of infinite nuclear matter. As a consequence, only gross features of the empirical potential, such as the behaviour of the energy dependence of its strength can be analysed, and from such an analysis one can expect to get information only on the order of magnitude of the range of nonlocality of the generalized optical potential.

2. Mathematical Problems

In this part of the report we analyse the mathematical problems arising both in the definition of the generalized optical potential and in the derivation of its basic properties such as the dispersion relation. We restrict ourselves to the nonrelativistic case, i. e. we assume that the many nucleons system is described by a nonrelativistic Schrödinger equation.

In such a case the generalized optical potential has been derived by many authors in the framework of formal scattering theory starting both from the time-dependent [3] and from the time-independent Schrödinger equation [1, 2, 4].

The expression given by FESHBACH [2] for the generalized optical potential seems to be the most convenient one in order to derive its basic properties.

Before analysing the mathematical problems involved in the proof of Feshbach's equation and in the derivation of the features of the generalized potential, we have to specific the notations we shall use.

Let us consider a system of $A + 1$ nucleons and, for the sake of simplicity, let us neglect spin and isospin variables and the identity of nucleons. In addition, let us suppose that the lowest channel (with threshold energy ε_1) be a channel with a free nucleon (incident nucleon) and a nucleus of A nucleons in the ground state Φ_A (target nucleus). We make no assumptions on the other channels with threshold energies ε_i ($i = 2, 3, \ldots$) greater than ε_1. In the energy range from ε_1 to ε_2 only the elastic scattering of the nucleon by the target nucleus is allowed and, for this reason, we denote the channel with threshold energy ε_1 as *elastic channel*. Besides,

we denote as *projection on the elastic channel* the projection operator P defined as follows:

$$P \, \Psi(\mathbf{r}, \mathbf{R}_A) = \Phi_A(\mathbf{R}_A) \, \psi(\mathbf{r}) \qquad (2.1)$$

where $\psi(\mathbf{r}) = \int \Phi_A{}^*(\mathbf{R}_A) \, \Psi(\mathbf{r}, \mathbf{R}_A) \, \mathrm{d}\mathbf{R}_A \quad (2.2)$

and $\mathbf{r}$ is the position of the incident nucleon relative to the center of mass of the target nucleus, whereas $\mathbf{R}_A$ denotes any set of internal coordinates of the nucleons in the target nucleus. If Ψ is the scattering wave function for the system of $A + 1$ nucleons, then ψ can be called the *optical model scattering wave function*. Finally we set $Q = 1 - P$.

Now, let the operator H be the Hamiltonian for the system of $A + 1$ nucleons in the center of mass system and let V_1 be the sum of all the interactions (two-body local potentials) between the incident nucleon and the nucleons in the target nucleus; then the operator $H_1 = H - V_1$ is the so-called channel Hamiltonian for the elastic channel and H_1 commutes with the projection operators P, Q. H_1 has a purely continuous spectrum in the interval $[\varepsilon_1, +\infty)$.

By means of the notations previously introduced, Feshbach's form for the generalized optical potential is the following:

$$\begin{aligned} V(E) \, \Psi &= P \, V_1 \, P \, \Psi \\ &\quad + P \, V_1 \, Q (E + i \, 0 - Q \, H \, Q)^{-1} Q \, V_1 \, P \, \Psi \\ &= P \, V_1 \, P \, \Psi + U(E) \, \Psi \, . \end{aligned} \qquad (2.3)$$

Here the generalized potential is written as an operator on the scattering wave function for the system of $A + 1$ nucleons; besides E is the total energy of the system of $A + 1$ nucleons and the notation $E + i \, 0$ denotes the limit as the imaginary part of E tends to zero for positive values. From Eq. (2.3) one can derive the form of the generalized potential as an operator on the optical model wave function ψ. Equation (2.2); the result is:

$$\mathcal{V}(E) \, \psi(\mathbf{r}) = \mathcal{V}_1(\mathbf{r}) \, \psi(\mathbf{r}) + \mathcal{U}(E) \, \psi(\mathbf{r}) \qquad (2.4)$$

where the local term $\mathcal{V}_1(\mathbf{r})$ corresponds to the term $P \, V_1 \, P$ in Eq. (2.3) and the operator $\mathcal{U}(E)$ is related to the operator $U(E)$ of Eq. (2.3) as follows:

$$\mathcal{U}(E) \, \psi(\mathbf{r}) = \int \Phi_A{}^*(\mathbf{R}_A) \, U(E) \, \Psi(\mathbf{r}, \mathbf{R}_A) \, \mathrm{d}\mathbf{R}_A \, . \tag{2.5}$$

From Eq. (2.3) it follows that the generalized optical potential depends on the energy only through the resolvent of the operator $Q \, H \, Q$. That is the great value of Eq. (2.3) because one can reduce the

mathematical analysis of the definition and of the properties of the generalized optical potential to an analysis of the spectral properties of the operator $Q H Q$. Indeed one can say that the whole Feshbach theory of nuclear reactions is proved if the following spectral properties of the operator $Q H Q$ are proved:

i) the continuous spectrum of $Q H Q$ is the interval $[\varepsilon_2, +\infty)$, whereas the point spectrum of $Q H Q$ in the interval $(-\infty, \varepsilon_2)$ contains only isolated eigenvalues of finite multiplicity and is bounded from below;

ii) the operator $Q H Q$ has a complete orthonormal family of eigenfunctions (including eigenfunctions associated with the points of the continuous spectrum).

Property i) is a sufficient condition in order to have a Hermitian generalized optical potential in the energy range from ε_1 to ε_2, i. e. in the energy range where only elastic scattering is allowed. Besides, by means of property ii), Feshbach proves that the operator $\mathcal{U}(E)$ in Eq. (2.4) is an integral operator and that, if we denote by $\mathcal{U}(E; \boldsymbol{r}, \boldsymbol{r}')$ its kernel, then the following dispersion relation holds:

$$\operatorname{Re} \mathcal{U}(E; \boldsymbol{r}, \boldsymbol{r}') = \sum_n \frac{A_n(\boldsymbol{r}, \boldsymbol{r}')}{E - \mathcal{E}_n}$$
$$+ \frac{1}{\pi} P \int_{\varepsilon_2}^{+\infty} \frac{\operatorname{Im} \mathcal{U}(E'; \boldsymbol{r}, \boldsymbol{r}')}{E' - E} \, dE' \qquad (2.6)$$

where the $\mathcal{E}_n$'s are the eigenvalues of $Q H Q$ and the kernels $A_n(r, r')$ have a finite rank as a consequence of the finite multiplicity of the eigenvalues of $Q H Q$.

The previously stated properties are the basic properties of the generalized optical potential and, at the same time, the starting point for Feshbach's unified theory of nuclear reactions. For instance one can give an interpretation of the eigenvalues of $Q H Q$ as resonances in the scattering of nucleons by nuclei [2].

Before discussing the validity of the spectral properties i), ii) we have to make some remarks about the operator $Q H Q$. Indeed, the spectral properties i), ii) can hold only if the operator $Q H Q$ is self-adjoint or if it has a unique self-adjoint extension. Now, under very general conditions on the two-body potentials, it has been proved [6] that the operator H is self-adjoint and lower semi-bounded with a domain $D_H \subset L^2(R^{3A})$ which can be completely characterized. Then one can easily verify that $P D_H \subset D_H$

and, as a consequence, that the operator $Q H Q$, with domain D_H, is symmetric and lower semi-bounded [7]. By Friedrich's theorem [8] it follows that the operator $Q H Q$ admits a self-adjoint extension, but it is not possible to prove, in the general case, that this extension is unique.

One can easily overcome the previous difficulty by introducing the operator:

$$\hat{H} = H - \varepsilon_1 P - P V_1 - V_1 P \qquad (2.7)$$

which commutes with the projection operator P and which satisfies the relation:

$$Q H Q \Psi = \hat{H} Q \Psi \qquad (2.8)$$

for any $\Psi \in D_H$. Indeed, as a consequence of Eq. (2.8) one can repeat Feshbach's derivation (or the derivation given in Ref. [9]) of the generalized optical potential with the operator $Q H Q$ replaced by the operator $\hat{H}$. On the other hand the analysis of the operator $\hat{H}$ is easier than the analysis of the operator $Q H Q$: if the two-body potentials acting between the nucleons satisfy rather general conditions (which include the Coulomb potential but exclude hard-core potentials), then one can prove [9] that the operators $P V_1$ and $V_1 P$ have a unique bounded extension; since the operator H is self-adjoint and lower semi-bounded, it follows that the operator $\hat{H}$ with domain D_H is also self-adjoint and lower semi-bounded.

As a consequence of the previous remark, we see that if the spectral properties i), ii) hold true for the operator $\hat{H}$ then Feshbach's theory is proved; in other words, it is not necessary that the spectral properties i), ii) hold true for the operator $Q H Q$. We remark also a typical feature of the generalized potential: if the operator $\hat{H}$ is replaced by the operator $\hat{H}' = \hat{H} + P T P$, where the operator T is arbitrary except for the restriction that $\hat{H}'$ has still to be self-adjoint, then the operator $\hat{H}'$ produces the same generalized potential as the operator $\hat{H}$, even if property i) may be false for $\hat{H}$.

Property i) has been proved for the operator $\hat{H}$ at least in the case of a three-nucleon system [7], with conditions on the two-body potentials which still include Coulomb potential but exclude hard-core potentials. In the proof methods introduced by FADDEEV [10] and HUNZIKER [11] are used. Of course it must be possible to extend the proof to the case of

an arbitrary number of nucleons, but the rather involved analytical methods used in the theory of many particle scattering [12, 13] seem to be necessary.

As a consequence of the previous results one can conclude that the generalized potential exists at least in the energy range from ε_1 to ε_2 and that it is a bounded (self-adjoint) operator.

As regards the spectral property ii), one can try to replace this property by von Neumann's result concerning the existence of a spectral measure associated with a self-adjoint operator [8]. Now, the existence of the spectral measure does not imply that the generalized optical potential is a bounded nonlocal potential in the energy range where inelastic channels are open; however the existence of the spectral measure implies that it satisfies a generalized dispersion relation [9] which, for the sake of conciseness, we do not write down here and which makes reasonable Eq. (2.6). Thanks to this result, it seems that it is not necessary to prove property ii) (which, of course, is more stringent than the existence of the spectral measure) but it is enough to prove that the resolvent of the operator H is an integral operator with a kernel satisfying suitable conditions. Results in this direction have not yet been obtained although they are customary in the theory of many particle scattering [13].

In order to complete the analysis of the mathematical problems arising in the theory of the optical model, we have still to analyse the connection between the generalized and the empirical optical potential.

The transition from the generalized to the empirical optical potential involves the following problems:

a) The generalized optical potential gives the scattering amplitude at a given energy while the empirical optical potential gives the average, on an energy interval, of the scattering amplitude. Therefore, starting from the same generalized potential different empirical potentials corresponding to different averaging intervals (which lead to the giant structure or to the intermediate structure of the cross section [14, 15] can be derived. However the problem of the average amplitudes becomes unimportant when energies are considered such that the fine and intermediate structures disappear. Since such a condition is also required for the validity of the approximation of infinite nuclear matter, in the following pages we do not consider this problem.

b) The generalized potential is nonlocal, while the empirical potential is local. Therefore the problem of the connection between a nonlocal potential and a local one which gives the same phase-shifts ("equivalent local potential") has to be considered. Such a problem has been widely analysed in recent years [16−22] and its full treatment involves the theory of the Schrödinger equation with a nonlocal potential [23]. However the problem becomes trivial when the nonlocal potential is rotationally and translationally invariant. Such a situation is appropriate for infinite nuclear matter and approximately holds in the interior of heavy nuclei for wave lengths of the incident nucleon small with respect to nuclear dimensions. If we write the nonlocal potential as follows [24]:

$$\mathcal{V}(E) \, \psi(\mathbf{r}) = \mathcal{V}_1 \psi(\mathbf{r}) + \int \mathcal{V}_2(|\mathbf{r} - \mathbf{r}'|) \, \psi(\mathbf{r}') \, d\mathbf{r}' \\ + \int \mathcal{U}(E; |\mathbf{r} - \mathbf{r}'|) \, \psi(\mathbf{r}') \, d\mathbf{r}' \quad (2.9)$$

(where $\mathcal{V}_1$ is a real constant corresponding to the local term in Equation (2.4); besides we have introduced explicitly the energy independent nonlocal term $\mathcal{V}_2$ which is due to the identity of the nucleons) and if we call $V_L(E)$ the equivalent local potential, the relation between the two potentials is:

$$V_L(E) = \mathcal{V}_1 + \mathcal{I}(E, k^2) \quad (2.10)$$

$$\mathcal{I}(E, k^2) = \int [\mathcal{V}_2(s) + \mathcal{U}(E; s)] \\ \times \exp\{i(\mathbf{k}, \mathbf{s})\} \, d\mathbf{s} , \quad s = |\mathbf{s}| \quad (2.11)$$

$$k^2 = \frac{2m}{\hbar^2} [E + V_L(E)]. \quad (2.12)$$

Equations $(2.10) - (2.12)$ are used also for relativistic energies of the incident nucleon with only a change in the kinematics. Indeed in that case Eq. (2.12) is replaced as follows:

$$k^2 = \frac{1}{\hbar^2 c^2} [E_t^2 - m^2 c^4 + 2 E_t V_L(E)], \\ E \gg |V_L(E)| \quad (2.13)$$

where E_t is the total relativistic energy.

Equations $(2.10) - (2.12)$ implicitly define the equivalent local potential.

3. Dispersion Relation Analyses of the Energy Dependence

For the comparison with experiment, we concentrate our attention on the energy dependence because this is characterized by the dispersion relation

(2.6) which is a property of fundamental character and the only one which is formulated in an explicit way by the theory of the generalized optical potential. It holds true also in the case of identical particles [2, 25]. In this way one obtains an understanding, at a fundamental level, of the energy dependence of the empirical potential which by itself does not satisfy the dispersion relation.

The experimental results [26-32] for neutrons and protons over various nuclei and over the energy range up to 1 GeV are shown in Figure 1 and 2. More detailed information about particular nuclei and for restricted energy intervals is summarized in Refs. [26, 27]. In these references are also summarized the various empirical formulae proposed to describe the energy dependence of the real part of the empirical potential.

The problem of the interpretation of this energy dependence is as old as the optical model itself. The same year of the work of FESHBACH, PORTER and WEISSKOPF [33] a description of the dispersive properties of the nuclear matter was proposed by BRUECKNER and coworkers [34] giving the correct order of magnitude of the potential well at zero energy by using the expression of the optical potential (in terms of the nucleon-nucleon forward scattering amplitude) developed by WATSON [1] at that time.

In 1956 FRAHN [35] showed that the equivalent local potential of a nonlocal energy independent potential in nuclear matter [only the term $\mathcal{V}_2$ in Eqs. (2.9) and (2.11)] has an energy behaviour which is in qualitative agreement with the experiment up to 300 MeV. The range of nonlocality (defined as the standard deviation of a gaussian distribution of nonlocality) was found to be .95 fm. A nonlocal energy independent potential of Frahn-Lemmer type [36] was then used by PEREY and BUCK [16] to fit the data of the neutron scattering on various nuclei up to 24 MeV. The value there obtained for the range of nonlocality was .85 fm. This work contains the first optical model analysis with a nonlocal potential, later applied by other authors [37, 38] and introduces also for finite nuclei an approximate equivalent local potential in a way similar to that used by Frahn for nuclear matter. It is remarkable that the numerical analysis shows that such potential reproduces quite well the data in spite of the fact that its derivation for finite nuclei contains some points which are not justified. An equivalent local

potential of this type was later used by ENGELBRECHT and FIEDELDEY [39] to obtain the energy dependence of the empirical potential starting from a nonlocal energy independent potential. While Perey and Buck considered only a surface nonlocal absorption, these authors, by introducing also a nonlocal volume absorption, obtain an excellent agreement with the experimental local real potential depth for neutrons on various nuclei up to 160 MeV with the same range of nonlocality already given by Perey and Buck.

It must be pointed out, however, that a local energy dependent potential derived from a nonlocal energy independent one cannot give any transition from attractive to repulsive interaction which is indicated by the phenomenological analyses around 300 MeV. Moreover, the model of an optical potential nonlocal and energy independent has no theoretical ground and can be expected to give agreement with experiment only over limited energy intervals. Indeed the nonlocal generalized optical potential has an intrinsic energy dependence, which comes from very fundamental reasons, such as causality [2, 40]. Therefore the problem handled by the more recent works [41-44] is, following an early suggestion by FESHBACH [2], to describe the energy dependence of the empirical potential as coming from two sources: one is the intrinsic energy dependence of the generalized optical potential ("causal" or "dynamical" energy dependence) and the second is its nonlocality ("spurious" energy dependence). The first, as due to causality, can be discussed by means of a dispersion relation. From such approach one hopes to obtain two results:

i) a theoretical prediction of the energy dependence of the real part of the strength of the empirical optical potential over a wide energy range;

ii) an estimate of the range of nonlocality of a suitable model describing the generalized optical potential.

A first indication on the "spurious" energy dependence of the empirical optical potential is obtained by applying directly the dispersion relation (2.6) to the data [25]. Such an application is not straightforward because: i) the empirical potential concerns the energy averaged scattering amplitude; ii) the energies of the resonances (eigenvalues of the operator $Q H Q$ or $\hat{H}$) and the residues at the poles should be known; iii) the asymptotic behaviour of

the imaginary part of the empirical potential at high energies, where the data are lacking, is needed.

Difficulty i) is overcome in Ref. [25] by working at energies above the resonances (see also the remarks at the end of Sect. 2), whereas difficulty ii) is overcome by considering only situations where the approximation of infinite nuclear matter holds (no poles). As regards iii), the unknown asymptotic behaviour of the imaginary part introduces only an unknown additive constant in the real part (at lower energies), since the contribution to the dispersion integral coming from the asymptotic region depends very weakly on the energy until this is rather low.

The result given in Ref. [25] and later on confirmed in Refs. [41−44] clearly shows, *in the energy range up to 400 MeV*, a disagreement between the slope of $\mathrm{Re}\, V_{\mathrm{L}}(E)$ calculated from the dispersion relation and the slope deduced from the experimental data. Indeed, while the empirical values of $\mathrm{Re}\, V_{\mathrm{L}}(E)$ decrease with the energy, the dispersion relation curve slowly increases up to 350 MeV. This comparison is shown in Fig. 1 where the dispersion relation curve [curve (a)], calculated by a subtracted dispersion relation from the curve (a) of Figure 2, has been normalized to the experimental values at high energy where its slope agrees with the experiment. An important "spurious" energy dependence of the empirical potential is then indicated, in qualitative agreement with the previous works which neglected the dynamical energy dependence [16, 35, 39].

The next step consists in separating the "dynamical" energy dependence from the "spurious" energy dependence and in expressing the latter in terms of the nonlocality of the generalized optical potential. The empirical optical potential is treated as the equivalent local potential of the generalized optical potential.

The various treatments are based on the use of the dispersion relation for $\mathcal{U}(E; s)$ and on an estimate of the function $J(E; k^2)$, − Eqs. (2.11), (2.12) − by means of suitable models for the s-dependence of $\mathcal{U}(E; s)$.

In Refs. [42, 43] it is assumed that the nonlocality is given only by the term $\mathcal{V}_2$ in Eq. (2.9), due to the identity of the nucleons, and the generalized optical potential, after averaging over a suitable energy interval [15], is written as:
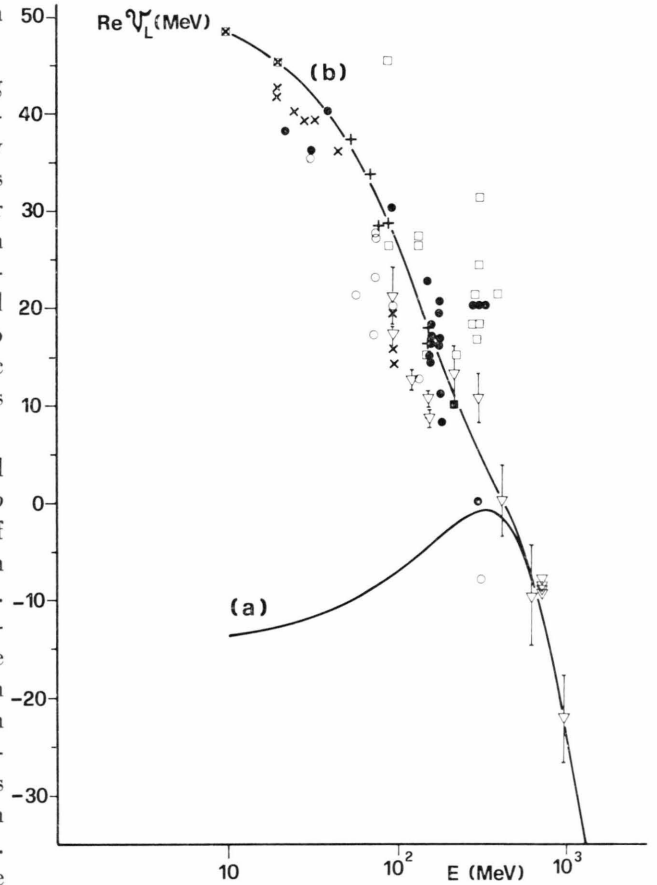

Fig. 1. Experimental values of the real part of the empirical optical potential well from Refs. [26−32]; + − neutrons, volume absorption; × − neutrons, surface absorption; ● − protons, volume absorption; ○ − protons, surface absorption; □ − protons, non-relativistic analyses; ▽ − protons, relativistic analyses; ⊕ − average, for neutrons and protons from Reference [40]. Curve (a): dispersion relation calculation; curve (b): dispersion relation calculation with the addition of the spurious energy dependence [41].

$$\mathcal{V}(E)\, \psi(\mathbf{r}) = \int \mathcal{V}_2(|\,\mathbf{r} - \mathbf{r}'\,|)\, \psi(\mathbf{r}')\, \mathrm{d}\mathbf{r}' \\ + \mathcal{U}(E)\, \psi(\mathbf{r}) \tag{3.1}$$

where $\mathcal{U}(E)$ is a local complex potential. As regards $\mathcal{V}_2$, Gaussian or Yukawian models are assumed:

$$\mathcal{V}_2(s) = U\, H_\beta(s) \tag{3.2}$$

where $\quad H_\beta(s) = (\pi\, \beta^2)^{-3/2} \exp\{-s^2/\beta^2\} \tag{3.3}$

or $\quad H_\beta(s) = (\pi\, \beta^2\, s)^{-1} \exp\{-2\, s/\beta\}. \tag{3.4}$

The parameter β is a measure of the nonlocality. In the case of a Gaussian model − Eq. (3.3) − if $(\beta^2\, m/2\, \hbar^2)\, |\,\mathrm{Im}\, V_{\mathrm{L}}(E)\,| \ll 1$, then by means of Eqs. (1.8) − (1.10) one gets:
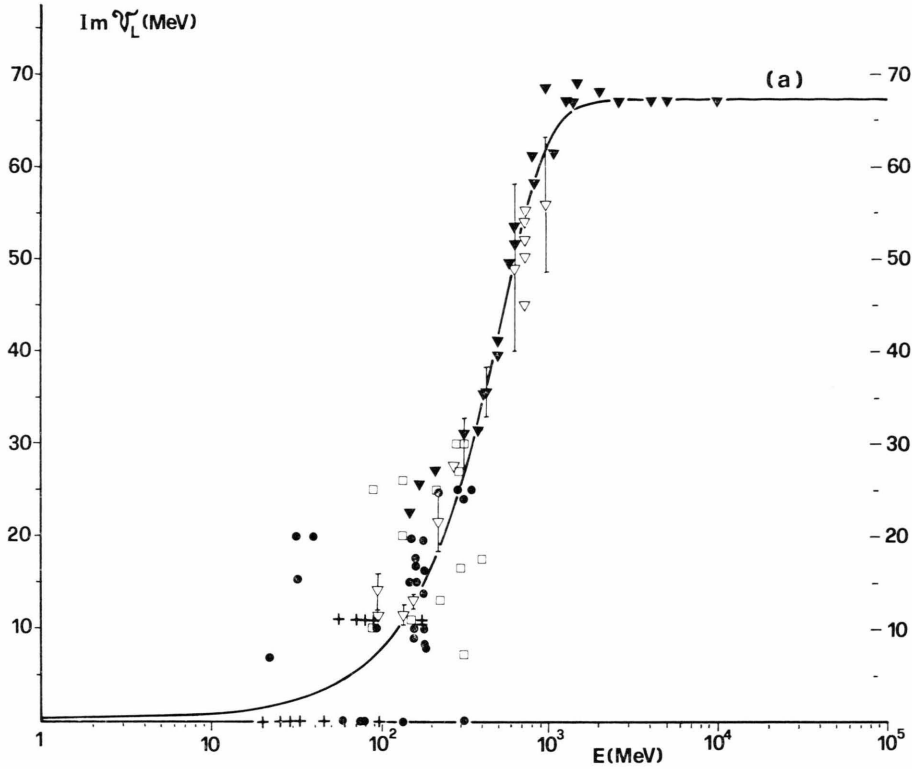
Fig. 2. Experimental values of the imaginary part of the empirical optical potential well. Curve (a): imaginary part used into the dispersion integral [25, 41].

$$\operatorname{Re} V_{\mathrm{L}}(E) = U \exp\left[-\frac{\beta^2 m}{2\hbar^2}\left(E + \operatorname{Re} V_{\mathrm{L}}(E)\right)\right] + \operatorname{Re} U(E) \tag{3.5}$$

$$\operatorname{Im} V_{\mathrm{L}}(E)\left\{1 + \frac{\beta^2 m}{2\hbar^2} U \exp\left[-\frac{\beta^2 m}{2\hbar^2}\left(E + \operatorname{Re} V_{\mathrm{L}}(E)\right)\right]\right\} = \operatorname{Im} U(E). \tag{3.6}$$

Equation (3.5) shows in a simple way the separation between the "spurious" energy dependence (first term at the r.h.s.) and the "dynamical" one (second term at the r.h.s.). This latter can be calculated by inserting the l.h.s. of Eq. (3.6) in the dispersion relation and it results as being made up by two terms: the first one contains in the dispersion integral only the imaginary part of the empirical optical potential; the second one is more complicated but under very reasonable approximations it can be neglected. In Ref. [42] the first term is calculated by means of the data for protons on ^{40}Ca, ^{12}C and ^{58}Ni up to about 200 MeV. In Ref. [43] it is assumed that $\operatorname{Im} V_{\mathrm{L}}(E)$ has a behaviour, obtained from the nonlocal energy-independent model of Reference [39]. As regards the asymptotic behaviour of $\operatorname{Im} V_{\mathrm{L}}(E)$, these authors, treating the optical potential as non-relativistic concept, make a non-relativistic extension of it to high energy only for its use in the dispersion relation. In Ref. [42] the extension is made by means of the formula

$$V_{\mathrm{L}}(E) = \frac{2\pi\hbar^2}{m}\varrho(0)\,f(E,0) \tag{3.7}$$

[where $\varrho(0)$ is the nucleon density at the center of the nucleus, normalized to the nucleon number, and $f(E,0)$ is the forward nucleon-nucleon scattering amplitude] and assuming for $f(E,0)$ a non-relativistic model; in Ref. [43] the empirical optical potential obtained from the nonlocal energy-independent potential of Ref. [39] is extrapolated at high energies. In both cases $\operatorname{Im} V_{\mathrm{L}}(E)$ tends to zero for increasing energy, but the high-energy behaviour remains quite arbitrary and this fact causes, at low energies

$(E \lesssim 200 \, \text{MeV})$ an arbitrary additive constant in $\text{Re}\,V_{\text{L}}(E)$. These works confirm the results already found in Ref. [25] on the importance of the "spurious" energy dependence of $V_{\text{L}}(E)$ at low energies. In Ref. [42], with the form (3.3) for H_β, values of β from 1.1 fm to 1.2 fm are obtained for the three nuclei considered; in Ref. [43] the value $\beta = 0.85$ fm is found in the case of the form (3.3) for H_β, while the value $\beta = 1.5$ fm is found in the case of the form (3.4) for H_β.

In Ref. [42] the dispersion relation is applied to the potential integrated over the nuclear volume and the Coulomb potential is taken into account. In Ref. [43], on the other hand, the radial dependence of the potential is discussed in detail with particular reference to the surface properties.

The points of view which characterize these treatments consist essentially in: a) assuming the energy dependent part of the generalized optical potential as local; b) treating the optical potential as a non-relativistic quantity whose imaginary part tends to zero for increasing energy. Point a) does not seem to be justified, at present, on the basis of the known representations of the generalized optical potential; point b) is not important at low energy (it causes only an additive constant) but prevents the application of this analysis to higher energies.

A different point of view is assumed in Ref. [41], where a relativistic meaning is assumed for the optical potential, as discussed in the introduction. The behaviour of the imaginary part of the empirical optical potential at high energies $(E \gtrsim 400 \, \text{MeV})$ is then calculated by means of the formula:

$$\text{Im}\,V_{\text{L}}(E) = \tfrac{1}{2}\,\hbar\,v_{\text{lab}}\,\gamma\,\varrho\,\bar{\sigma} \tag{3.8}$$

(where γ is the Goldberger factor accounting for the exclusion principle [45] and $\bar{\sigma}$ is the average nucleon-nucleon experimental total cross-section).

It is remarkable that, *beyond 400 MeV*, the experimental data for $\text{Re}\,V_{\text{L}}(E)$ are still strongly energy dependent and the slope is in good agreement with the dispersion relation calculation (as it is shown in Figure 1). This fact may be interpreted as indicating that the generalized optical potential contains, at least at these energies, a local energy-dependent term satisfying the dispersion relation and giving the – now completely "dynamical" – energy dependence of the empirical optical potential. However it must be emphasized that this fact does not mean at all that the generalized optical

potential becomes local: the contribution of the nonlocal terms of the empirical optical potential through the function $\mathfrak{I}(E, k^2)$ in Eqs. (2.11), (2.12) is averaged out, because the wave-length has become small in comparison to the range of nonlocality.

These ideas can be easily formulated by writing the Eq. (2.10) in the form:

$$V_{\text{L}}(E) = \mathcal{V}_1 + \mathcal{U}_1(E) + \mathfrak{I}(E, k^2) \tag{3.9}$$

where $\mathcal{U}_1(E)$ is a local complex potential, and by calculating $\text{Re}\,\mathcal{U}_1(E)$ by means of the dispersion relation. There are however some ambiguities in the choice of $\text{Im}\,\mathcal{U}_1(E)$. At high energy $(E \geqq 400 \, \text{MeV})$ Eqs. (3.9) and (2.11) show that $\text{Im}\,\mathcal{U}_1(E) \cong \text{Im}\,V_{\text{L}}(E)$; therefore Eq. (3.8) has been used in the asymptotic region for the calculation of $\text{Im}\,\mathcal{U}_1(E)$. At lower energies $\text{Im}\,\mathcal{U}_1(E)$ has been assumed as a function increasing from zero to the value given by Eq. (3.8) at 400 MeV. The insensitivity of the final result to the details of this behaviour has also been tested. Besides, since in this way $\text{Im}\,\mathcal{U}_1(E)$ goes asymptotically to a constant, the dispersion relation has been used in the subtracted form and the additive constant has been determined by matching $\text{Re}\,V_{\text{L}}(E)$ to the high energy data $(E \cong 1 \, \text{GeV})$.

Below 400 MeV the term $\mathfrak{I}(E, k^2)$ increases for decreasing energy and accounts for the discrepancy between $\text{Re}\,\mathcal{U}_1(E)$ and the experimental $\text{Re}\,V_{\text{L}}(E)$. It mainly describes the spurious" energy dependence and it has been simulated by a Gaussian form factor with two parameters (well depth and nonlocality); therefore Eq. (3.9) is written in the following form:

$$\begin{aligned} \text{Re}\,V_{\text{L}}(E) = \mathcal{V}_1 &+ \text{Re}\,\mathcal{U}_1(E) \\ &+ N \exp\left\{ -\tfrac{1}{4}\,\beta^2\,\text{Re}\,k^2 \right\}. \end{aligned} \tag{3.10}$$

The parameter N is determined by the low energy data $(E \cong 20 \, \text{MeV})$ while β is determined by the general behaviour of the experimental points [see curve (b) on Figure 1], obtaining a value of 0.8 fm.

The Eq. (3.10) for $\text{Re}\,V_{\text{L}}(E)$ gives a very good agreement with the data in the whole energy range from 20 MeV up to 1 GeV. Differing from the other analyses considered here, the change of the sign of the optical potential well around 300 MeV is reproduced. Unfortunately, the data above 400 MeV, which are crucial in order to test this model, are few and in part refer to light nuclei. The logarithmic asymptotic behaviour indicated in this work, due to the subtracted dispersion relation calculation

of the term $\mathcal{U}_1(E)$ in Eq. (3.10), does not agree, however, with an optical model analysis at 21 GeV [46], which gives a much smaller value for the (repulsive) real part of the empirical potential.

Some features of the previous work, such as the relativistic point of view and the nonlocal character of the energy dependent part of the generalized optical potential, are also assumed in a paper worked out quite independently and at the same time [44]. Here the following formula, already given in Ref. [47].

$$V_L(E) = \frac{K_0(E) - (m/3\,\hbar^2)\,E\,K_2(E)}{1 + (m/3\,\hbar^2)\,K_2(E)} \qquad (3.11)$$

is assumed, where

$$K_n(E) = \int s^n [\mathcal{V}_2(s) + \mathcal{U}(E;s)]\,ds\,,$$
$$(n = 0, 1, 2, \ldots). \qquad (3.12)$$

This assumption amounts to taking up only the first two terms in the expansion of the function $\mathcal{I}(E, k^2)$ in powers of k^2:

$$\mathcal{I}(E, k^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,k^{2n}\,K_{2n}(E). \qquad (3.13)$$

If $K_0(E)$ is expressed by means of the dispersion relation for $\mathcal{U}(E;s)$ integrated over s, and if $K_2(E)$ is assumed to vanish above 1 GeV, then from Eq. (3.11) one gets the subtracted dispersion relation

$$\mathrm{Re}\,V_L(E) - \mathrm{Re}\,V_L(E_0) = \frac{E - E_0}{\pi}\,P \int_0^{+\infty} \frac{\mathrm{Im}\,V_L(E')}{(E'-E)\,(E'-E_0)}\,dE' + E_0\,\alpha(E_0) - E\,\alpha(E) \qquad (3.14)$$

where the function $\alpha(E)$ is approximated by a two-parameter rational function

$$\alpha(E) = A/(E+B). \qquad (3.15)$$

The dispersion relation (3.14) is a three parameter formula for $\mathrm{Re}\,V_L(E)$ like Eq. (3.10) used in Ref. [41] and reproduces in a very satisfactory way the general features of the behaviour of $\mathrm{Re}\,V_L(E)$ in the whole energy range covered by the experimental data with the experimental value for $\mathrm{Re}\,V_L(E_0)$ $(E_0 = 50\,\mathrm{MeV})$ and $A = 78,3\,\mathrm{MeV}$, $B = 195,7\,\mathrm{MeV}$. These last values agree also with estimates for the effective mass. Equation (3.14) is remarkable because it is simple and independent of models on the coordinate dependence of the nonlocal part of the generalized optical potential. Moreover, in the dispersion integral only a directly measurable quantity is implied, while in Ref. [41] the function $\mathrm{Im}\,\mathcal{U}_1(E)$ is required.

The success of the simple parametrization shown in Eq. (3.15) may perhaps signify that it expresses in a simplified way analytical properties of an approximate equivalent local potential, even if some points in its derivation do not appear justified, as keeping only the first two terms in the power expansion (3.13), assuming that $K_2(E)$ wanishes at high energies and, more generally, that the generalized optical potential can be considered as local for increasing energy in spite of the energy independent nonlocal term $\mathcal{V}_2(s)$ due to the antisymmetrization. Such requirement is quite different from assuming the existence of a local energy de-

pendent term (or of a term tending to a local potential for increasing energy), as is made in Ref. [41].

The formulation of the generalized optical potential in terms of a multiple scattering expansion [1,48,49] strongly suggests the existence of such a term. The question is critically discussed in Ref. [49] where it is shown that this fact certainly holds for the one-pion exchange contribution to the generalized optical potential in the non-relativistic approximation. The relativistic extension requires a careful study of the off-shell nucleon-nucleon scattering amplitude. In Ref. [49] it is also shown that, if the off-shell nucleon-nucleon scattering amplitude depends only on the momentum transfer (the conditions for the validity of such a hypothesis are not quite clear), then a local term exists in the generalized optical potential and is given by Eq. (3.7). If this result is true, such a term can be identified at high energies with the empirical optical potential, because the contribution to this one, coming from the nonlocal term, is averaged out at these energies [41].

It must be noted that the identification proposed here has a meaning completely different from the "local approximation" which is made, also at low energies, on the single-scattering term (of the multiple scattering expansion for the generalized optical potential) by inserting the momentum of the incident nucleon *inside* the nucleus in place of the momenta before and after the scattering [50] (see also [47], p. 795 and [48], p. 150). This approximation

amounts to transforming the nonlocal potential into a local one, according to Eqs. (2.10) − (2.12) and that is the reason of the agreement between such approximation and the empirical data [51], in spite of the large nonlocality of the generalized optical potential.

We finally observe that the papers previously reviewed, although developing different points of view, agree in finding a dominant „spurious" energy dependence at low energies ($E \lesssim 200$ MeV). Besides the treatments developing a relativistic point of view can account for the change of sign of the real part of the empirical optical potential around 300 MeV. It is also indicated that, although the generalized optical potential maintains its nonlocal character, at least owing to the Pauli principle, the nonlocal effects on the energy dependence of the empirical optical potential vanish when the wave length of the incident nucleon becomes small enough. Therefore, at high energies ($E \gtrsim 500$ MeV) the energy dependence of the empirical optical potential should be completely dynamical and given by the local part of the generalized optical potential. As a consequence it seems important to concentrate the atten-

tion on this one and such an analysis implies a careful knowledge of the properties of the off-shell nucleon-nucleon scattering amplitude.

In our opinion, analyses of this type are important mainly for giving information on general features rather than on details of the optical potential, as a consequence of the crude schematization there involved. Therefore a careful choice of the data does not seem so important but rather the possibility of disposing of a lot of experimental information, mainly on heavy nuclei and over any energy range. Data above 200 MeV are, at present, very scarce. Measurements in this range should be very interesting, since the region around 400 MeV seems to make the transition from the dominance of the "spurious" part to the dominance of the "dynamical" part in the energy dependence of the empirical optical potential. This fact, when confirmed, should give a quantitative information on the nonlocality of the generalized optical potential and an indication about the energy range where the empirical optical potential can be described in a simple way in terms of nucleon-nucleon scattering amplitudes.

[1] K. M. WATSON, Phys. Rev. **89**, 575 [1953]; **105**, 1388 [1957].
[2] H. FESHBACH, Ann. Physics **5**, 357 [1958]; **19**, 287 [1962].
[3] F. COESTER and H. KÜMMEL, Nucl. Phys. **9**, 225 [1958].
[4] A. K. KERMAN, H. McMANUS, and R. M. THALER, Ann. Physics **8**, 551 [1959].
[5] H. FESHBACH, A. GAL, and J. HÜFER, Ann. Physics **66**, 20 [1971].
[6] T. KATO, Trans. Amer. Math. Soc. **70**, 195 [1951].
[7] M. BERTERO, Nuovo Cim. **2 A**, 605 [1971]. In this paper the operator $Q H Q$ has been considered as self-adjoint.
[8] K. YOSIDA, Functional Analysis, Springer-Verlag, Berlin 1965. Chapter XI.
[9] M. BERTERO and G. PASSATORE, Nuovo Cim. **2 A**, 579 [1971].
[10] L. D. FADDEEV, Mathematical Aspects of the Three-Body Problem in the Quantum Theory of Scattering, Israel program for Scientific Translations, Jerusalem 1965.
[11] W. HUNZIKER, Helv. Phys. Acta **39**, 451 [1966].
[12] O. A. YAKUBOVSKY, Sov. J. Nucl. Phys. **5**, 937 [1967].
[13] K. HEPP, Helv. Phys. Acta **42**, 425 [1969].
[14] H. FESHBACH, A. K. KERMAN, and R. H. LEMMER, Ann. Physics **41**, 230 [1967].
[15] R. LIPPERHEIDE, Z. Physik **202**, 58 [1967].
[16] F. PEREY and B. BUCK, Nucl. Phys. **32**, 97 [1962].
[17] F. PEREY and D. S. SAXON, Phys. Lett. **10**, 107 [1964].
[18] W. E. FRAHN, Nucl. Phys. **66**, 358 [1965].
[19] N. AUSTERN, Phys. Rev. **137**, B 752 [1965].
[20] H. FIEDELDEY, Nucl. Phys. **77**, 149 [1966]; **A 96**, 463 [1967]; **A 115**, 97 [1968].
[21] M. COZ, L. G. ARNOLD, and A. D. MACKELLAR, Ann. Physics **58**, 504 [1970]; **59**, 219 [1970].
[22] F. CAPUZZI, Nuovo Cim. **11 A**, 801 [1972].

[23] M. BERTERO, G. TALENTI, and G. A. VIANO, Nuovo Cim. **46 A**, 333 [1966]; Comm. Math. Phys. **6**, 128 [1967]; Nucl. Phys. **A 113**, 625 [1968]; **A 115**, 395 [1968]; Nuovo Cim. **62 A**, 27 [1969].
[24] Hereafter we call potential the well depth (i. e. the potential changed in sign) as is usually done in the papers on the optical model.
[25] G. PASSATORE, Nucl. Phys. **A 95**, 694 [1967].
[26] P. E. HODGSON, The Optical Model of the Elastic Scattering, Clarendon Press, Oxford 1963.
[27] P. E. HODGSON, Nuclear Reactions and Nuclear Structure, Clarendon Press, Oxford 1971.
[28] P. G. McMANIGAL, R. D. EANDI, S. N. KAPLAN, and B. J. MOYER, Phys. Rev. **137**, B 620 [1965].
[29] C. J. BATTY, Nucl. Phys. **23**, 562 [1961].
[30] L. ROSEN, J. C. BERRY, A. S. GOLDHABER, and E. H. AUERBACH, Ann. Physics **34**, 96 [1965].
[31] P. G. ROOS and N. S. WALL, Phys. Rev. **140**, B 1237 [1965].
[32] T. A. ERIKSON and G. A. RINANDER, Ark. Fys. **33**, 377 [1967].
[33] H. FESHBACH, C. E. PORTER, and V. F. WEISSKOPF, Phys. Rev. **96**, 448 [1954].
[34] K. A. BRUECKNER, C. A. LEVINSON, and H. M. MAHMOUND, Phys. Rev. **95**, 217 [1954].
[35] W. E. FRAHN, Nuovo Cim. **4**, 313 [1956].
[36] W. E. FRAHN and R. H. LEMMER, Nuovo Cim. **5**, 1564 [1957].
[37] D. WILMORE and P. E. HODGSON, Nucl. Phys. **55**, 673 [1964].
[38] P. C. SOOD, Nucl. Phys. **84**, 106 [1966].
[39] C. A. ENGELBRECHT and H. FIEDELDEY, Ann. Physics **42**, 262 [1967].

[40] R. LIPPERHEIDE, Nucl. Phys. **89**, 97 [1966].
[41] G. PASSATORE, Nucl. Phys. **A 110**, 91 [1968].
[42] R. LIPPERHEIDE and A. K. SCHMIDT, Nucl. Phys. **A 112**, 65 [1968].
[43] H. FIEDELDEY and C. A. ENGELBRECHT, Nucl. Phys. **A 128**, 673 [1969].
[44] I. AHMAD and M. Z. RAHMAN KHAN, Nucl. Phys. **A 132**, 213 [1969].
[45] M. L. GOLDBERGER, Phys. Rev. **74**, 1269 [1940].
[46] W. E. FRAHN and G. WIECHERS, Ann. Physics **41**, 442 [1967].

[47] M. L. GOLDBERGER and K. WATSON, Collision Theory, John Wiley & Sons, New York 1964.
[48] A. L. FETTER and K. M. WATSON, Adv. Theor. Physics **1**, 115 [1965].
[49] R. E. SCHENTER and B. W. DOWNS, Phys. Rev. **133 B**, 522 [1964].
[50] R. M. FRANK, J. L. GAMMEL, and K. M. WATSON, Phys. Rev. **101**, 891 [1956].
[51] A. E. GLASSGOLD, Rev. Mod. Phys. **30**, 419 [1958].

Mach's Principle — A Critical Review [§]

MICHAEL REINHARDT

Astronomische Institute der Universität Bonn

After a short historical introduction it is discussed how far Mach's principle is incorporated into general relativity. The possible rôle of Mach's principle as a selection rule for the solutions of Einstein's field equations is summarized. Then follows a discussion of Mach's principle in theories of gravitation other than Einstein's, mainly the Brans-Dicke theory. Finally the experiments on the isotropy of inertial mass and their consequence for Mach's principle are described. The conclusion is that Mach's principle, though an extremely stimulating thought, has at present little claim to be a basic physical principle.

"They sought it with thimbles, they sought it with care;
They pursued it with forks and hope;
They threatened its life with a railway share;
They charmed it with smiles and soap."
LEWIS CARROLL: *"The Hunting of the Snark"*

I. Introduction

Mach's principle in its most general form can be stated as: *The inertial mass of a body is caused by its interactions with the other bodies in the universe.*

Despite, or perhaps because of, its vagueness and elusiveness it has intrigued many physicists and exerted a considerable influence on fundamental physics. It has played an important, if only heuristic, rôle in the construction of theories of gravitation, from Einstein's general relativity up to the Brans-Dicke theory. Nevertheless there is no consensus among physicists as to its standing in physics. The range of opinions extends from complete negation of the postulate, as being either physically irrelevant, or wrong, to the whole-hearted agreement in the sense that any theory of gravitation must comply

with Mach's principle in order to be acceptable. The present paper tries to give a concise account of the concrete results which emerged in the long history of this controversy.

And its history is long indeed. It goes right back almost to the very beginnings of physics. NEWTON [1] himself asked the question of what singled out the inertial frames in which his first law holds. From his famous experiment of the rotating water pail [1,2] he concluded that, since the shape of the water surface is independent of the motion of the bucket relative to the surface, it is not rotation with respect to other matter that determines the appearance of centrifugal and Coriolis forces but rotation relative to absolute space, which exists a priori and independently of the bodies in it. Thus it is acceleration relative to absolute space, or better its absence, that determines the preferred inertial frames. Leibniz, based on other philosophers back to Plato, rejected the idea of an absolute space [3]. He hold the relational view that "spatium est ordo coexistendi". For a discussion of the different notions on space hold by Newton and Leibniz see e. g. NORTH [4]. EULER [5] took the Newtonian point of view and influenced Kant's attitude [4], which in turn dominated the nineteenth century until Mach's critique of Newtonian mechanics. Mach had however an early precursor in